Saturation momentum at fixed and running QCD coupling

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Abstract

A relationship, linking the saturation momentum in the case of fixed and running QCD coupling, respectively, is derived from the Balitsky-Kovchegov equation. It relies on the linear instability of the evolution equation in the dilute regime. The leading orders of the saturation momenta are mapped onto each other exactly. For subleading terms a qualitative correspondence is achieved with a relative error going to zero for large rapidities. The relationship can also be derived for the Balitsky-Kovchegov equation with a cutoff accounting for low-density effects and is satisfied by the corresponding isoline functions. Further implications arise for the existence of travelling-wave solutions in the two situations.

I. INTRODUCTION AND RESULTS

In deep-inelastic scattering the total cross-section σ for the scattering of a virtual photon with momentum q on a proton with momentum p is a function of the virtuality $Q^2 = -q^2$ and the rapidity $Y = \ln(1/x)$ with $x = Q^2/(2p \cdot q)$. At sufficiently small $x \lesssim 10^{-2}$ the cross-section σ becomes a function of the ratio of the virtuality Q^2 and a function ${Q_s}^2(Y)$ of rapidity Y called saturation momentum: $\sigma = \sigma[Q^2/{Q_s}^2(Y)]$ [1]. This phenomenon is called geometric scaling. Translated to the dipole scattering amplitude N, it becomes a function of the difference between the logarithm L of the square of the momentum variable conjugate to the dipole size and the logarithm of the saturation momentum: $N = N[L - \ln {Q_s}^2(Y)]$.

In quantum chromodynamics (QCD) the scattering of dipoles is described by the Balitsky hierarchy [2]. In the factorising limit it reduces to the Balitsky-Kovchegov (BK) equation [3]. In the low-density regime it in turn simplifies to the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [4].

In Ref. [5] scaling solutions for the BK equation are identified at fixed coupling and with the second order (”diffusive”) approximation [17] for the BFKL kernel. That investigation is based on knowledge about the existence of travelling-wave solutions for the Fisher-Kolmogorov-Petrovsky-Piskunov (FKPP) equation [6].

This approach is not feasible for the running coupling BK equation because it belongs to a different universality class. Different methods have to be used, like a rescaling of the rapidity Y [7], introduction of curved absorptive boundaries into the BFKL equation [8], or use of a travelling-wave ansatz [9] [18]. Note that, in general, the ”time” and ”space” variables need not be linear functions of the rapidity Y and the momentum variable L.

Here a mapping between the saturation momentum at fixed and at running coupling, respectively, is derived. The derivation is based on the observation that in the scaling regime the saturation momentum characterises an isoline of the amplitude N. It is derived by neglecting commutators of the BFKL kernel and the running-coupling function acting on the amplitude N. This procedure is justified by the fact that the BK equation is linearly unstable with respect to small perturbations around $N = 0$ (pulled front). At this level of accuracy it is shown to hold for the saturation momentum in both cases at leading order for large rapidities Y. For the subleading terms the qualitative features are reproduced: They provide a negative but small correction to the leading behaviour. The relative error connected to the quantitative deviations vanishes for large rapidities Y.

As said instability is already present for the BFKL equation the relationship also holds there. Actually the validity of the relation seems to be widely independent of the detailed form of the equation of motion and the form of the running coupling function's equivalent. It appears to hold for a larger class of differential equations. In this context, the linear instability seems to be a sufficient condition. The aforementioned quantitiative deviations are non-universal in the sense that they depend on details of the equation of motion which are independent of whether or not it is linearly unstable.

Furthermore, the relationship between the isolines for fixed and running coupling can also be derived after a cutoff has been introduced into the growth term of the BK equation in order to accomodate effects beyond the mean-field approximation [10, 11, 13]. Even if, strictly speaking, the thus modified BK equation is no longer linearly unstable against arbitrarily small perturbations around $N = 0$, the leading-order terms for large rapidities Y are again mapped onto each other exactly. The subleading terms decay exponentially [12]. For the approximation made in the course of the relationship's derivation to hold, it suffices that the BK equation with cutoff is unstable for sufficiently large perturbations.

The comparison of isoline plots [14] obtained by solving the BK equation numerically [15] for fixed and running coupling respectively, according to the relations presented below, would also provide interesting insights and cross checks.

In section II the mapping between the isolines for the BK equation for fixed and running coupling, respectively, is derived. In section III the validity of the relationship is checked for the leading (subsection III A) and sub-leading

terms (subsection III B) of the saturation momentum. In subsection III C the reason for the accuracy of the relation is discussed. It is explained how it can be generalised to a larger class of differential equations. Section IV treats the mapping for the BK equation with a cutoff taking into account low-density effects. For convenience appendix A exposes details of the connection between the fixed coupling BK equation and the FKPP equation.

II. RELATION

The BK equation for the dipole forward-scattering amplitude as a function of the rapidity Y and the momentum variable L is given by:

$$\frac{\partial N}{\partial Y} = \bar{\alpha}\left[\chi\left(-\frac{\partial}{\partial L}\right) N - N^2\right] \tag{1}$$

with the BFKL kernel:

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \tag{2}$$

where $\psi(\gamma)$ is the digamma function—the logarithmic derivative of the gamma function—and where the BFKL eigenvalue-function with a differential operator in the argument is defined via its series expansion around $\gamma_0 \in]0;1[$.

The phenomenon of scaling manifests itself by the isolines of the amplitude $N(Y, L)$ keeping their distance from each other in the direction of the variable L constant if the rapidity Y changes. In other words, the amplitude is only a function of the difference $L - \ln Q_s(Y)^2$. Consequently, in the scaling regime, the saturation momentum $Q_s = Q_s(Y)$ characterises the isoline with $N[Y, Q_s(Y)] = N_s$ [19]. There, the equations describing two isolines differ merely by an additive constant.

Independent of the phenomenon of scaling, any isoline $L_i = L_i(Y)$ of the amplitude $N(Y, L)$ satisfies the relation

$$\frac{dL_i}{dY} = -\frac{\partial N}{\partial Y}\left(\frac{\partial N}{\partial L_i}\right)^{-1}. \tag{3}$$

For fixed QCD coupling, $\partial N/\partial Y$ can been replaced by the right-hand side of (1).

$$\frac{dL_f}{dY} = -\bar{\alpha}\left[\chi\left(-\frac{\partial}{\partial L_f}\right) N_f - {N_f}^2\right]\left(\frac{\partial N_f}{\partial L_f}\right)^{-1}. \tag{4}$$

The running coupling case is obtained through the replacement $\bar{\alpha} \to (bL)^{-1}$:

$$\frac{dL_r}{dY} = -\frac{1}{bL_r}\left[\chi\left(-\frac{\partial}{\partial L_r}\right) N_r - {N_r}^2\right]\left(\frac{\partial N_r}{\partial L_r}\right)^{-1}, \tag{5}$$

The constant b is linked to the QCD-β-function and reads: $b = (11n_c - 2n_f)/(12n_c)$. n_c stands for the number of colours and n_f for the number of (massless) flavours.

Hence, division of Eq. (4) by L_f, taking into account the common initial condition

$$N_f(Y_0, L) = N_0(L) = N_r(Y_0, L), \tag{6}$$

yields:

$$\frac{1}{L_f}\frac{dL_f}{dY_0} = \frac{dL_r}{dY_0}. \tag{7}$$

The constants $\bar{\alpha}$ and b have been omitted for the sake of clarity. It suffices to keep in mind to exchange the two in a comparison.

Analogous relations for the higher derivatives can be derived by repetition of the above steps: After taking the derivatives of Eqs. (4) and (5) with respect to the rapidity Y, replace the new occurrences of the derivative $\partial N/\partial Y$ on the right-hand side with the help of the BK equation (1). Subsequently, divide the expression obtained from Eq. (4) by L_f. This allows to identify the respective right-hand sides at $Y = Y_0$ up to terms involving commutators of the BFKL kernel with the running coupling function acting on the amplitude $[\chi(-\partial_L), L^{-1}]_- N$. They are going to be omitted in what follows. Why this is justified is going to be investigated in subsection III C.

Together with the initial condition:

$$L_f(Y_0) = L_0 = L_r(Y_0) \tag{8}$$

this hierarchy of equations can be summarised by:

$$\left[\frac{1}{L_f(Y_0)}\frac{d}{dY_0}\right]^n L_f(Y_0) = \left[\frac{d}{dY_0}\right]^n L_r(Y_0) \tag{9}$$

for all $n \in \mathbb{N}_0$. This defines all Taylor coefficients of $L_r(Y)$ at rapidity $Y = Y_0$ based on those of $L_f(Y)$.

Inversely, by multiplying with L_r each time instead of dividing by L_f one obtains all Taylor coefficients of $L_r(Y)$ from those of $L_f(Y)$:

$$\left[\frac{d}{dY_0}\right]^n L_f(Y_0) = \left[L_r(Y_0)\frac{d}{dY_0}\right]^n L_r(Y_0). \tag{10}$$

Equation (9) can be reexpressed as:

$$L_r(Y_0 + \delta Y) = \exp\left\{\delta Y \frac{1}{L_f(Y_0)}\frac{d}{dY_0}\right\} L_f(Y_0) \tag{11}$$

and Eq. (10) accordingly as:

$$L_f(Y_0 + \delta Y) = \exp\left\{\delta Y L_r(Y_0)\frac{d}{dY_0}\right\} L_r(Y_0) \tag{12}$$

for all δY. This can be verified by Taylor expansions around $\delta Y = 0$. The exponentials in Eqs. (11) and (12) are operators for conformal mappings. In the following calculations the interpretation as translation operators is to be used. (Alternative computations could, for example, involve dilatation operators.) With the definitions:

$$\frac{dZ_f}{dY} = L_f(Y), \tag{13}$$

and

$$\frac{dZ_r}{dY} = \frac{1}{L_r(Y)}, \tag{14}$$

respectively, these turn into:

$$\begin{aligned} L_r(Y_0+\delta Y) &= \exp\left\{\delta Y \frac{d}{dZ_f}\right\} L_f[Y_f(Z_f)] = \\ &= L_f\{Y_f[Z_f(Y_0)+\delta Y]\} \end{aligned} \tag{15}$$

and

$$\begin{aligned} L_f(Y_0+\delta Y) &= \exp\left\{\delta Y \frac{d}{dZ_r}\right\} L_r[Y_r(Z_r)] = \\ &= L_r\{Y_r[Z_r(Y_0)+\delta Y]\} \end{aligned} \tag{16}$$

for all δY. $Y_f(Z_f)$ and $Y_r(Z_r)$ are the inverse functions of $Z_f(Y)$ and $Z_r(Y)$, respectively, as defined in Eqs. (13) and (14). Eqs. (15) and (16) provide a direct link between the isolines of the solutions for the equations of motion in the fixed and the running coupling case. They hold irrespective of whether the exact or an approximative expression, for example, the second-order expansion around $\gamma = \gamma_c$, is used for the BFKL kernel (2).

III. COMPARISON

For large rapidities Y, the universal expression for the (logarithm of the) saturation momentum—up to an arbitrary additive constant—in the case of fixed coupling available in the literature reads [8, 9, 16]:

$$L_f(Y) = \bar{\alpha}\frac{\chi(\gamma_c)}{\gamma_c}Y - \frac{3}{2\gamma_c}\ln Y - \frac{3}{{\gamma_c}^2}\sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)}}\sqrt{\frac{1}{Y}} \tag{17}$$

The BFKL kernel (2) has been expanded around $\gamma_0 = \gamma_c$ which solves the equation $\gamma_c\chi'(\gamma_c) = \chi(\gamma_c)$ and has the numerical value $\gamma_c = 0.6275...$ [9]. The third term is only given in [16] and there the error is of the order $\mathcal{O}(Y^{-1})$. For the running coupling case the two leading terms are known [8, 9]:

$$L_r(Y) = \sqrt{\frac{2\chi(\gamma_c)}{b\gamma_c}Y} + \frac{3}{4}\left[\frac{\chi''(\gamma_c)}{\sqrt{2b\gamma_c\chi(\gamma_c)}}\right]^{\frac{1}{3}}\xi_1 Y^{\frac{1}{6}} \tag{18}$$

with $\xi_1 = -2.338...$ being the rightmost zero of the Airy function Ai. The universally valid expression, i.e., independent of the initial conditions, can at most include terms that decrease more slowly than $Y^{-1/2}$. Like in the case of fixed QCD coupling, this can be checked through a shift ΔY of the rapidity Y in the above equation and subsequent expansion around $\Delta Y = 0$ [16]. Thus, in general, the rapidity variable Y in the last two equations differs by this additive constant ΔY.

A. Leading order

Starting out with the first term in Eq. (17) one finds from Eq. (13):

$$Z_f(Y) = \bar{\alpha}\frac{\chi(\gamma_c)}{2\gamma_c}Y^2. \tag{19}$$

Inversion yields:

$$Y_f(Z_f) = \sqrt{\frac{2\gamma_c}{\bar{\alpha}\chi(\gamma_c)}Z_f} \tag{20}$$

Evaluation at $Z_f(Y_0) + \delta Y$ leads to:

$$Y_f[Z_f(Y_0)+\delta Y] = \sqrt{\frac{2\gamma_c}{\bar{\alpha}\chi(\gamma_c)}[Z_f(Y_0)+\delta Y]}. \tag{21}$$

Replacing the rapidity Y in the first term of Eq. (17) by the right-hand side of the previous expression and afterwards $\bar{\alpha}$ by b^{-1} yields:

$$\sqrt{\frac{2\chi(\gamma_c)}{b\gamma_c}(Y_0+\delta Y+\Delta Y)} = L_r(Y_0+\delta Y), \tag{22}$$

where constants have been absorbed in ΔY. Thus, to this order, Eq. (15) is satisfied. With analogous calculations also Eq. (16) can be verified to leading order.

B. Subleading terms

With the first two terms in Eq. (17) one finds from Eq. (13):

$$Z_f(Y) = \bar{\alpha}\frac{\chi(\gamma_c)}{2\gamma_c}Y^2 + \frac{3}{2\gamma_c}(1-\ln Y)Y, \tag{23}$$

the exact inverse of which cannot be given analytically. The factor $(1-\ln Y)$ varies slowly as compared to powers of Y. Regarding it as fixed in every step, one can determine the inverse iteratively. Starting out with $\ln Y_f^{(0)}(Z_f) = 1$ leads to Eq. (20) for $Y_f^{(1)}(Z_f)$. Replacing $\ln Y$ in the logarithm in Eq. (23) by the preliminary result $\ln Y_f^{(1)}(Z_f)$ [in general $\ln Y_f^{(n)}(Z_f)$], subsequent inversion, and selection of the positive solution in every step finally leads to the recursion relation:

$$\begin{aligned} Y_f^{(n+1)}(Z_f) = -\frac{3\left[1-\ln Y_f^{(n)}(Z_f)\right]}{2\bar{\alpha}\chi(\gamma_c)} + \\ + \sqrt{\left\{\frac{3\left[1-\ln Y_f^{(n)}(Z_f)\right]}{2\bar{\alpha}\chi(\gamma_c)}\right\}^2 + [Y_f^{(1)}(Z_f)]^2} \end{aligned} \tag{24}$$

It shows that the dominant behaviour for large Z_f—and hence large $Y_f^{(1)}(Z_f)$—can be obtained after a finite number of iterations. The dominant terms are given by:

$$Y_f^{(n+1)}(Z_f) = Y_f^{(1)}(Z_f) + \frac{3}{2\bar{\alpha}\chi(\gamma_c)} \ln Y_f^{(n)}(Z_f) - \\ -\frac{1}{2}\left[\frac{3}{2\bar{\alpha}\chi(\gamma_c)}\right]^2 \frac{[\ln Y_f^{(n)}(Z_f)]^2}{Y_f^{(1)}(Z_f)} + \mathcal{O}\left[\frac{1}{Y_f^{(1)}(Z_f)}\right], \quad (25)$$

where constant terms have been omitted. Replacing the rapidity Y in the first two terms of Eq. (17) by the right-hand side of the previous expression yields:

$$\bar{L}_r = \bar{\alpha}\frac{\chi(\gamma_c)}{\gamma_c} Y_f^{(1)}(Z_f) - \frac{9}{8}\frac{1}{\bar{\alpha}\chi(\gamma_c)\gamma_c}\frac{[\ln Y_f^{(1)}(Z_f)]^2}{Y_f^{(1)}(Z_f)} + \\ +\mathcal{O}\left[\frac{\ln Y_f^{(1)}(Z_f)}{Y_f^{(1)}(Z_f)}\right], \quad (26)$$

which can already be obtained from the second-order result, i.e., from Eq. (24) with $n = 1$. Evaluation at $Z_f(Y_0) + \delta Y$ leads to the replacement:

$$Y_f^{(1)}(Z_f) \to Y_f^{(1)}[Z_f(Y_0) + \delta Y] = \\ = \frac{\gamma_c}{\bar{\alpha}\chi(\gamma_c)}\sqrt{\frac{2\chi(\gamma_c)}{b\gamma_c}(Y_0 + \delta Y + \Delta Y)}, \quad (27)$$

which coincides with Eq. (21). Finally, $\bar{\alpha}$ has to be replaced by b^{-1}.

Again, the leading term of Eq. (18) is reproduced. The first subleading terms in Eqs. (18) and (26) together with (27) do not coincide exactly. However, they are similar qualitatively. Through their inclusion with the leading term, L_r and $\bar{L}_r$ are both diminished. The relative error vanishes for large rapidities Y like: $(\bar{L}_r - L_r)/(\bar{L}_r + L_r) \sim Y^{-1/3}$.

In principle, it is possible to base the above comparison on Eq. (16). While the integral required for solving Eq. (14) is still known analytically for $L_r(Y)$ given by Eq. (18), the inversion of the resulting expression is more cumbersome than for Eq. (23).

C. Discussion

As demonstrated in the previous subsection, neglecting the commutator $[\chi(-\partial_L), L^{-1}]_{-}N$ leads to a mapping exact at leading order and with subleading deviations whose relative error vanished for large rapidities. In what follows it shall be discussed why this is the case.

Omitting said commutator is equivalent to approximating the prefactors of the m^{th} derivatives with respect to the momentum variable L occuring on the running-coupling side during the relationship's derivation by the term dominant for large L: $[L^{-1} + \mathcal{O}(L^{-2})]\partial_L{}^m N \approx L^{-1}\partial_L{}^m N$. Subsequently one would have to justify why L is effectively so large that the above steps are feasible. One is tempted to bring forward the fact that saturation physics is protected from the influence of the infrared, i.e., from small L. However, through the repetition of the identical steps the above relationship can also be derived for the BFKL equation and in the BFKL equation no saturation effects are encoded.

As this line of arguments is not conclusive let us investigate how shifting the BFKL kernel by an additive constant $\bar{\chi}(\gamma) = \chi(\gamma) + \delta$ does affect the expression for the saturation momentum. First for arbitrary values of the shift $\delta \in \mathbb{R}$ the modified critical value $\bar{\gamma}_c$ for the argument γ, defined through $\bar{\chi}(\bar{\gamma}_c) = \bar{\gamma}_c\bar{\chi}'(\bar{\gamma}_c)$ obeys $0 < \bar{\gamma}_c < 1$. Therefore the solution stays always in the supercritical regime of the FKPP equation $\bar{\gamma}_c^{-1} > 1$ [20], i.e., it has always a universal travelling-wave solution [5, 9]. Hence one can explore the effect of the shift δ directly with the help of the expression for the saturation momentum in Eq. (17).

For $\delta > -4\ln 2$, $\bar{\chi}(\gamma)$ remains positive definite and the same qualitative asymptotic behaviour is retained, although reaching the asymptotic regime might require extremely large rapidities Y if $\bar{\chi}(\bar{\gamma}_c) \ll 1$.

The picture changes for $\delta = -4\ln 2$, where $\bar{\gamma}_c = \frac{1}{2}$ and $\bar{\chi}(\bar{\gamma}_c) = 0$, i.e, the minimum of the kernel becomes zero. The term proportional to the rapidity Y is absent. As explained above, by shifting the BFKL kernel one stays in the supercritical regime of the FKPP equation. However, the L and the x axes are not parallel to each other. In this particular situation $x \sim 2t$ is mapped exactly onto $L =$ const. (see Appendix A), whence the linear term of the saturation momentum vanishes although the FKPP equation has a supercritical travelling-wave solution. In other words, for $\delta = -4\ln 2$ the BK equation is not linearly unstable for small perturbations around $N = 0$ even if the FKPP equation is.

The effect on the expression for running coupling (18) is more drastic. While the term proportional to the square root of the rapidity Y vanished in unison with the linear term in the fixed coupling case, the prefactor of the second addend diverges like $\sim \bar{\chi}(\gamma_c)^{-1/6}$. Hence, in the running-coupling case the previous description breaks down altogether.

Proceeding to $\delta < -4\ln 2$, leads to $\bar{\chi}(\gamma)$ also taking negative values and especially to a negative critical value $\bar{\chi}(\bar{\gamma}_c) < 0$. Thus the term in the fixed-coupling saturation momentum proportional to the rapidity Y reappears but with a negative sign. This means that for growing rapidities Y the amplitude decreases. With this kernel the BK equation is stable against perturbations around $N = 0$. In this range, according to Eq. (18), the saturation momentum in the running-coupling case would even be complex. This shows that the latter case would have to be investigated anew.

Summarising, the term of the differential equation important for the instability around $N = 0$ and hence for the mapping to work is the one originating from the critical value of the kernel $\chi(\gamma_c)$. Preserving only this term in

the BK equation and obtaining the relationship between the isolines for the different couplings leads exactly to the previous results, because in this limit the crucial commutator vanishes $[\chi(\gamma_c), L^{-1}]N = 0$. The more general derivation in section II resums additional terms which lead to the same overall behaviour.

As an outlook, based on the discussions in this subsection the present approach should work for any pair of differential equations which are linearly unstable around the dilute state, widely independent of the details of the remaining terms and the detailed structure of the equivalent of the running coupling-constant. The main step for adapting to another situation is replacing the right-hand sides of Eqs. (14) or (13), respectively, by the new running coupling function $\bar{\alpha}(L)$ or its reciprocal $[\bar{\alpha}(L)]^{-1}$, respectively.

IV. LOW-DENSITY EFFECTS

The BK equation describes the mean-field limit of the Balitsky hierarchy. The mean-field approximation is best satisfied in the dense regime and least in the dilute. There fluctuations are important. As mentioned above, for the relevant boundary conditions, the BK equation describes the propagation into a linearly unstable state. This leads to a high sensitivity of the solution to modifications at the toe of the front. In Refs. [10, 13] it has been demonstrated that the principal correction to the front propagation speed can be simulated in a deterministic manner by cutting off the growth term for small values of the amplitude. For example in the diffusive approximation to the BK equation this leads to the following modified equation of motion:

$$\frac{\partial N}{\partial Y} = \bar{\alpha}\left[\chi_2 \frac{\partial^2 N}{\partial L^2} + \chi_1 \frac{\partial N}{\partial L} + \left(\chi_0 N - N^2\right) c(N)\right] \quad (28)$$

with the replacement $\bar{\alpha} \to (bL)^{-1}$ for the case of running coupling and where the coefficients $\chi_i,\ i \in \{1, 2, 3\}$ are given in Appendix A and with the cutoff function:

$$c(N) = \theta\left(N - {\alpha_S}^2\right). \quad (29)$$

Note that $1/{\alpha_S}^2$ equals the number of dipoles at saturation wherefore ${\alpha_S}^2$ gives the corresponding step height.

Carrying out the steps of the derivation beginning with Eq. (3) but this time for the modified BK equation (28) instead of its standard form (1) yields again Eqs. (15) and (16). Looking at the modified expressions for the saturation momentum one sees that Eqs. (15) and (16) are satisfied exactly: At fixed coupling the leading term of the isoline equation reads [10, 11, 13]:

$$L_f(Y) = \bar{\alpha}\left[\frac{\chi(\gamma_c)}{\gamma_c} - \frac{\pi^2}{2}\frac{\gamma_c \chi''(\gamma_c)}{\ln^2(1/{\alpha_s}^2)}\right] Y, \quad (30)$$

with exponentially small subleading terms [12]. At running coupling one finds [11]:

$$L_r(Y) = \sqrt{\frac{2}{b}\left[\frac{\chi(\gamma_c)}{\gamma_c} - \frac{\pi^2}{2}\frac{\gamma_c \chi''(\gamma_c)}{\ln^2(1/{\alpha_s}^2)}\right] Y}. \quad (31)$$

The introduction of the cutoff removes the linear instability of the BK equation with respect to perturbations smaller than the threshold ${\alpha_S}^2$. However, the instability for perturbations larger than ${\alpha_s}^2$ is sufficient to ensure that the modified expressions for the saturation momentum are qualitatively similar to those of the unmodified version. Consequently the mapping still works.

In this last context the term "saturation momentum" is avoided on purpose because after the inclusion of stochastic effects the solutions of the BK equation does no longer provide the observable amplitude but the amplitude for the scattering on a given partonic realisation of the target [11, 13]. The physical amplitude is obtained by means of an ensemble average [13] accounting for the non-vanishing variance of the front position [10]. Then the physical amplitude does no longer show geometric scaling [11, 13].

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APPENDIX A: MAPPING: FKPP↔BK

The fixed-coupling BK equation for $N = N(Y, L)$ in the second-order ("diffusive") approximation:

$$\frac{\partial N}{\partial Y} = \bar{\alpha}\left[\chi_2 \frac{\partial^2 L}{\partial L^2} + \chi_1 \frac{\partial N}{\partial L} + \chi_0 N - N^2\right], \quad (A1)$$

with:

$$\begin{aligned} \chi_0 &= {\gamma_c}^2 \chi''(\gamma_c)/2, \\ \chi_1 &= \gamma_c \chi''(\gamma_c) + \chi(\gamma_c)/\gamma_c, \\ \chi_0 &= \chi''(\gamma_c)/2, \end{aligned} \quad (A2)$$

is mapped onto the FKPP equation for $u = u(t, x)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2, \quad (A3)$$

by the relation:

$$N(Y, L) = [{\gamma_c}^2 \chi''(\gamma_c)/2] \times u[t(Y), x(Y, L)], \quad (A4)$$

with:

$$\begin{aligned} t &= \bar{\alpha} Y \times [{\gamma_c}^2 \chi''(\gamma_c)/2], \\ x &= \bar{\alpha} Y \times [{\gamma_c}^2 \chi''(\gamma_c) + \chi(\gamma_c)] + L \times \gamma_c. \end{aligned} \quad (A5)$$

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[17] The use of the approximate equation's solution is reasonable everywhere outside the deeply saturated regime. In the latter it goes to a constant while the solution of the exact equation continues to grow logarithmically [3].
[18] All of these approaches work also in the fixed-coupling case.
[19] The last condition defines this isoline also outside the scaling regime, although, strictly speaking, there, Q_s does not deserve the name "saturation momentum"
[20] In order to clarify the necessary connections Appendix A displays the mapping that leads to the FKPP equation.